Article #6

Basic Reasoning Skills

Basic reasoning skills are those processes basic to cognition of all forms. There are four categories of basic reasoning skills: (1) storage skills, (2) retrieval skills, (3) matching skills, and (4) execution skills.

Storage and retrieval skills enable the thinker to transfer information to and from long-term memory. These are the encoding strategies discussed in Chapter 6. The learner does something on purpose to focus on the information being studied or to relate it to information that is already in long-term memory. An example of a commonly used storage and retrieval technique is *visual imagery mediation*. The learner purposely develops a visual (or auditory, kinesthetic, or emotional) representation for the information to be remembered. Mnemonic strategies are also examples of storage and retrieval skills.

Matching skills enable a learner to determine how incoming information is similar to or different from information already stored in long-term memory. There are five types of matching skills:

Categorization enables learners to classify objects or ideas as belonging to a group and having the characteristics of that group. This has been referred to as *chunking* in chapter 6. It speeds up the thinking process, making it possible to generalize and to go beyond the information immediately given by the isolated object or idea.

When you look at an animal and call it a cat, you are categorizing. When you listen to a comedian and decide that a particular story was a stupid joke, you are categorizing. Any time you classify something as being an example of something you already know, you are categorizing. In the sense that it is used here, a categories are synonymous with concepts.

You will see a similarity between categorization and Piaget's concepts of organization, assimilation, and accommodation, which were discussed in Chapter 4.

Extrapolation enables learners to match the pattern of information from one area to that found in another area. This strategy assists the thinking process by making it unnecessary to start from scratch when learners encounter new information. Instead, the learner takes information that already exists for a different purpose and adapts it to a new situation.

If you know the basic rules of soccer but know nothing about rugby, you could extrapolate a great deal of your knowledge of soccer to help you understand rugby. If you have an understanding of the causes of the American revolution, you can extrapolate this information to help you understand the developments in the former Soviet Union in the early 1990's. Any time you take previous information and incorporate it into an understanding of a new topic, you are extrapolating. In the sense that it is used here, a extrapolation is synonymous with generalization.

Analogical reasoning involves seeing the similarities among essentially different objects or ideas and using existing knowledge about the first set of objects or ideas to understand the others. For example, a computer-literate person reading Chapter 6 of this book might realize that the short-term memory is similar to random access memory (RAM) and that long-term memory is similar to a hard drive. By using this analogy, the person would have a basis for understanding short-term memory, long-term memory, and the relationship between them.

Analogical reasoning enables learners to combine the first two basic reasoning processes (categorization and extrapolation) in order to deal with new information and new relationships more effectively.

A very large number of programs that train students to improve their thinking skills include an analogical reasoning component. In addition, tests that attempt to measure the thinking abilities of students often include an analogies section.

Evaluation of logic is the process of comparing the structure of information with an internalized system of logic to see if the information is valid or true. For example, students can learn to follow the rules of deductive and inductive logic and to look for and avoid specific types of errors, such as hasty generalizations and non sequiturs.

For many years, the study of logic has focused on formal reasoning - for example, the evaluation of syllogisms, the use of Boolean operators, and the use of Venn diagrams. While formal logic is important, recent research has begun to emphasize that human beings more commonly use other forms of thinking when they have to make decisions or solve problems - such as looking for analogies or being influenced by various biases. (These other forms of thinking are referred to as *informal logic* or *everyday thinking*.) Therefore, it may be more useful to help learners develop better analogies and to avoid sources of bias than to teach them formal logic. This is not because teachers should encourage students to be illogical, but because most errors arise before the person even gets to the point of trying to use formal logic.

Evaluation of value is the process of matching information to an internalized value system and analyzing the logic of that value system. For example, a learner might decide that a concept or a solution to a problem represents "the way things should be" and accept it as accurate. Or a person might realize that a certain piece of information (e.g., the exact names of the people in an anecdote) is not really worth remembering.

These value judgments often incorporate the motivational and affective aspects of learning - described in Chapters 5 and 8 of this book. The importance of these apparently non-cognitive aspects of thinking should not be underestimated.

Executive procedures are the final set of basic reasoning skills. These skills are *executive* in the sense that they coordinate a set of other skills in order help learners build new cognitive structures or drastically restructure old ones. (They act much like the executives in corporations, who coordinate the activities of other employees in order to achieve commercial goals.) There are three basic executive skills:

Elaboration is the process of inferring information not explicitly stated in what the learner saw or heard. Learners use such skills as categorization, elaboration, analogical reasoning, and information retrieval to make these inferences. For example, in the previous discussion of matching information to value systems, I made reference to attitudinal and motivational components of learning. I did not present in that paragraph a description of exactly what I meant by these terms - you had to infer that information. For example, I didn't tell you why a person's attitude would influence his judgment of its value. You had to figure that out for yourself, and you probably did so without effort. If you made a good inference, you had a good chance of understanding what I was talking about. If you made an incorrect inference, you probably missed the point of that paragraph. I tried to help you by writing as clearly as possible and by referring to chapters where the information is explained in detail.

To take another example, imagine yourself in the audience when Jesus Christ first told the parable of the Good Samaritan. The story is actually quite brief, but listeners would go well beyond the story itself. They would realize the enormity of the gap between the Samaritans and the Jews. They would realize that Jesus was putting the Samaritan on a level higher than the priests of their own religion. They would realize that the concept of neighbor that Jesus was using was vastly different from the one they had learned about. The parable doesn't state much of this explicitly; the listeners had to elaborate to have an effective understanding of this parable.

There are two reasons why elaboration is necessary: (1) the learning situation (book, teacher, problem setting, etc.) may provide incomplete information, or (2) the learner may not perceive all the information that is available. Neither of these reasons necessarily represents a "mistake." If learners are capable of elaboration, then both teachers and learners should take advantage of this phenomenon - and most teachers and learners do so automatically. Teachers skip details that learners can easily infer - if they didn't, their presentations would become unduly long and boring. Likewise, learners do not attend to every detail of a presentation; they focus on important details and infer others. {That is why proofreading is a different task than reading. Good readers do not focus on every letter in every sentence they read. They catch the important ideas and fill in the rest, because they know it is there.}

Good learners make good inferences regarding what they need to fill in. On the other hand, some learners make incredibly inaccurate inferences, and this leads to learning problems. Students who make bad inferences can become much better thinkers by learning to make better inferences.

Problem solving is the process of finding information or a strategy to achieve a goal &emdash; to overcome an obstacle. In school, the goal is usually to find declarative or procedural information in a content area. For example, a student may want to know the capital of South Dakota or how to calculate the actual cost of a house that he could buy for $80,000 with a 25-year loan at 9% interest. In life outside the classroom, the goal may be to overcome any sort of obstacle.

Almost everything a learner does can be viewed as directed toward solving a problem (Anderson, 1985). Problem solving has been described in many ways, but it usually consists of describing the problem, determining the desired outcome, selecting possible solutions, choosing strategies, testing trial solutions, evaluating the outcomes of these trials, and revising steps as necessary.

Problem solving is an important process that is described in detail in Newell & Simon (1972), Chipman, Segal, & Glaser (1985), Gagne (1985), Chance (1986), Lesgold (1988), Perkins & Salomon (1989), Gagne, Yekovich, & Yekovich (1993). Since the solution to problems often requires original thinking, *creativity* is often an important aspect of problem solving. Since it is important to evaluate the quality of solutions, *critical thinking* is often an important aspect of problem solving. Creativity and critical thinking are discussed later in this chapter.

Composing is the process of creating new information to express an idea. It can be viewed as a specific type of problem solving, in which the problem is to communicate ideas in an appropriate way to achieve a goal. Composing can consist of either written or oral communication of ideas. Although composition skills are often taught in English or language arts classes, they are employed in all areas of the curriculum. For example, social studies students may use their composing skills to integrate their ideas regarding the causes of the American Civil War or the progress of the human rights movement.

The composing process is discussed in greater detail in the Language Arts section of Chapter 16.

Executive skills are much like the metacognitive skills, which are discussed later in this chapter. The basic reasoning skills are summarized in Figure 7.1.

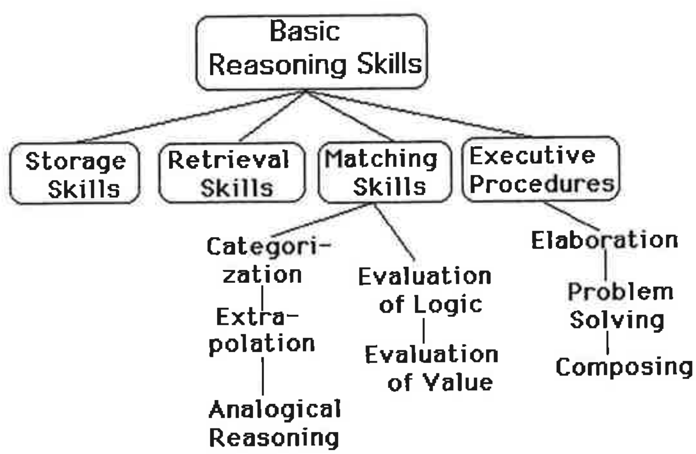


Figure 7.1. The Basic Reasoning Skills.

Click on a topic from the following list, or use your web browser to go where you want to go:

- Introduction
- Learning-to-Learn Skills
- Content Thinking Skills
- Basic Reasoning Skills <You are here>>
- Summary of Thinking Skills

Article #7

February 16, 2011

What is Problem Solving?

There are two major types of problem solving – reflective and creative. Regardless of the type of problem solving a class uses, problem solving focuses on knowing the issues, considering all possible factor and finding a solution. Because all ideas are accepted initially, problem solving allows for finding the best possible solution as opposed to the easiest solution or the first solution proposed.

What is its purpose?

The process is used to help students think about a problem without applying their own pre-conceived ideas. Defining what the problem looks like is separated from looking at the cause of the problem to prevent premature judgment. Similarly, clarifying what makes an acceptable solution is defined before solutions are generated, preventing preconceptions from driving solutions. Some people argue that problem solving is the art of reasoning in its purest form. In the classroom, problem solving is best used to help student understand complex ethical dilemmas, think about the future or do some strategic planning.

How can I do it?

Reflective Problem Solving follows a series of tasks. Once you have broken the students into groups, the students define the problem, analyze the problem, establish the criteria for evaluating solutions, propose solutions and take action.

Define the Problem: List all the characteristics of the problem by focusing on the symptoms, things affected, and resources or people related to defining the problem. In the end, pair down the thinking to a clear definition of the problem to be solved.

Analyze the Problem: Use the evidence you collected in step one to decide why the problem exists. This step is separate from defining the problem because when the steps are done together it is possible to prejudge the cause.

Establish Criteria: Set a clear objective for the solution. If the problem is too hard, break the objectives into two categories – musts and wants. Don't discuss solutions yet, just what criteria a solution must meet.

Propose Solutions: Brainstorm as many different solutions as possible. Select the one that best meets the objectives you stated as a part of the criteria for a solution.

Take action: Write a plan for what to do including all resources you will need to complete the plan. If possible, implement the plan.

Creative Problem Solving uses the same basic focus, but the process is less geared towards solutions and more towards a focus on brainstorming. The focus is on creating ideas rather than solving a clear existing problem. Sometimes the problem is pre-defined, and the group must focus on understanding the definition rather than creating it.

Orientation: Similar to defining the problem, orientation also focuses on being sure the group is prepared to work together. The group might take the time to agree upon behaviors or ways of saying things in addition to setting the context and symptoms of the issues. The group generates a series of headings that group the topics they must address.

Preparation and Analysis: Decide which headings are relevant or irrelevant. The group focuses on similarities and differences between ideas and works on grouping them into like categories. The group asks how and why a lot, and focuses on the root cause of the problem in a way that is similar to analyzing the problem.

Brainstorm: The group generates as many potential solutions as possible. At this point, all ideas are considered to be good ones.

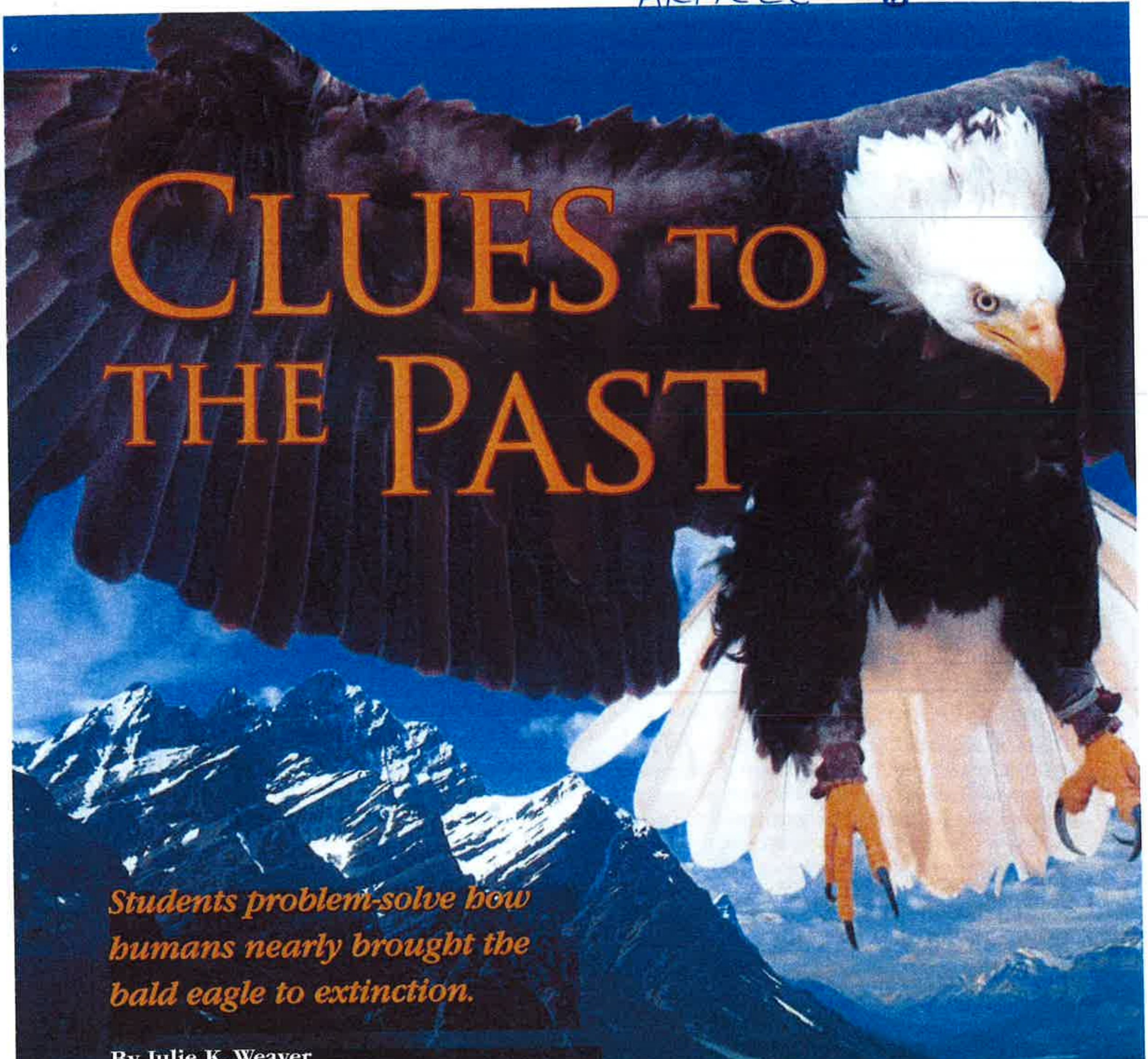
Incubation: Before deciding which solution is the best, the group should leave the problem for as much time as reasonable. Often several days or a week is ideal depending on the ages of the students. Leave enough time to develop distance but not long enough for students to lose the gist of their earlier work.

Synthesis and Verification: Start by establishing the criteria for a good solution, then look at all the brainstormed solutions and try to combine them to create the solution with the greatest numbers of positives and the smallest numbers of negatives.

How can I adapt it?

If you are working in a multi-grade room or on a project that involves a diverse group, problem solving is a great process for achieving consensus. You can also use parts of the process to help students challenge set thinking patterns.

ARTICLE #8



Clues to the Past

Students problem-solve how humans nearly brought the bald eagle to extinction.

By Julie K. Weaver

Students love a mystery. So what do America's most majestic bird, a bag of habitat clues, and a soft-shelled egg have in common? This easy-to-do activity engages students as they connect clues to problem-solve how the bald eagle reached the brink of extinction in the 1960s in the lower 48 states. Students arrive in my fourth-grade classroom with a basic foundation of life cycles, food chains, and habitats; however, their understanding of how these concepts are related is minimal. I designed this two- to three-day inquiry activity to give students an opportunity to wonder, discuss, and discover interactions in habitats.

My primary objective is for students to problem-solve how human use of a chemical can affect the environment. They'll identify and describe interactions in habitats, highlighting the diversity of the animals affected. The activity also allows me to assess prior knowledge, spot misconceptions, and evaluate how well students can explain relationships in habitats in words (both spoken and written). Writing plays an important role in the learning process. My students keep a science notebook, which I have modeled the use of, and our writing and data-recording activities build throughout the year. This notebook is an essential tool for the students to reflect on and demonstrate their learning.

the shell membrane. When compared to a regular egg, this instantly grabs the attention of the students as they consider the consequences of such thin eggshells. I assign the students the task of imagining they are a team of scientists in the 1960s and the U.S. government has just commissioned them to determine what is causing the bald eagle population to decline. I implore the groups to work quickly, as time is running out for the eagle.

Engagement: Mysterious Decline

I begin by asking the students why the bald eagle is appropriate as our country's national symbol. Elementary students easily come up with a list of many of the eagle's awesome traits—adaptations for hunting prey that include sharp talons, powerful beak, and keen eyesight. However, most are unaware of how close America came to losing eagles in the 1960s. To set the stage, I provide students with a timeline of some basic facts that include the following:

- 1782: The bald eagle is named America's national symbol.
- 1940: Bald Eagle Protection Act passes, prohibiting the taking, possession, and commerce of bald eagles due to the declining population.
- 1960s: Scientists report 417 nesting pairs of eagles remain in the lower 48 states. Scientists report few young eaglets are hatching. Further study finds the eggshells are thin. An analysis of eggshells reports traces of a chemical called *DDT*. The U.S. government commissions scientists to find out why the bald eagle is laying eggs with shells that are too thin and soft to hatch.

It might seem as though the answer is in the timeline, but for my students, it was just enough to get them curious and engaged in finding the connection.

I bring in an egg I have soaked in vinegar for 24 hours, which has lost its hard outer shell but can be held by

Exploration: Teacher-Directed Inquiry

Each cooperative-learning group is given a bag of clues from the eagle's habitat to piece together a possible reason for the population decline. Instructions are printed on the outside of the bag. My students are familiar with reading instructions and carrying out investigations with the support of their cooperative-learning group. I have found several benefits in this type of teacher-directed inquiry. First, students must activate their prior knowledge to make connections among the items in the bag. Then, together they must organize the clues, hypothesize, listen, and discuss possible interactions. They also have the opportunity to write in their science notebooks.

Instructions: Your team of scientists must determine what happened to the bald eagle based on the clues in the bag. They all fit together to tell the story. Put the clues together and come up with a logical story line. Time is of the essence; the eagle numbers are dropping fast. List of clues: river, raindrops, insect, small fish, farmer's crop, large fish, plankton, DDT.

I use fishing lures (without hooks) for the fish, plastic insects, and plastic plankton (Figure 1, p. 50). I cut the river and raindrops out of blue construction paper. The crop is corn or wheat grains in a resealable plastic bag. Check for food allergies before bringing food into the classroom. Be sure to rinse out the soda cans and place duct tape over any sharp edges.

CAUTION

I create and attach a label for DDT to a soda can for each student kit (see Internet Resources and NSTA Connection). This label has enough information for the students to understand the purpose of using DDT—to kill insects that damage crops.

Most groups begin by observing and discussing the contents of the bag. As they review the items, they begin to propose connections and test ideas within their group. I watch and listen to the groups hypothesize ideas and explore possible relations between the clues. I learn much about the students' prior knowledge of food chains, life cycles, and human interaction in environments. Two typical first hypotheses begin with either (a) the spilling or dumping of DDT into rivers by careless people or (b) insects sprayed by DDT being eaten by eagles. As the groups discuss and debate these possibilities, they realize that not all the clues can be accounted for in either scenario. At this point, many groups ask if they can gather more information about eagles to settle arguments and to clear up misconceptions about what eagles eat.

I provide books that contain basic information about eagles from our school library, being careful that the students use the chapters that do not have information about how DDT affected eagles. Students may also use the internet to find information, but I monitor the websites carefully (see Internet Resources). I have only two computers in my classroom available for the students to use, so I can easily monitor use of the websites. I do not want students to read about the movement of DDT through the habitat yet, but rather I want them to hypothesize and come up with their own explanations. As the clues are rearranged, each group begins to solve the environmental mystery. At this time, many students begin to record facts that will be important to their explanation in their science notebook.

Frequently, while students are problem-solving, I will announce that time is running out for the bald eagle; students must work quickly or the few nesting pairs left will die of old age without being able to reproduce. Their group will be called to Washington, D.C. soon to report their findings to the government. By the end of the first day, most groups have focused on a story line that connects the clues, and typically several groups are mostly correct at this point. Students naturally turn to their notebooks (an established daily habit) to record their ideas and visualize their thinking. I use a student-centered notebook in which students have been taught to use a combination of diagrams and written text to explain their thinking. They use it daily in class to help them think on paper. At times, I specifically ask for an entry in paragraph form, and other times, I request a diagram.

Figure 1.

Contents of the bag of clues.

PHOTOGRAPH COURTESY OF THE AUTHOR

Explanation: Soft-Shelled Eggs Revealed

On the second day, the groups organize their clues by laying them out in order on the laboratory table. Each group has a speaker who will present the predicted solution to the class. I provide the speaker with 10 minutes to practice prior to presenting to the class; this results in coherent explanations. Each group's speaker uses the clues organized on the table to help focus their presentation on the relation between the clues and the story that they have recorded in their science notebook. Students can interject facts gathered about eagle adaptations that add support to their presentation: how they hunt, where they nest, and how they raise their young.

I do not correct misconceptions, but listen, make positive comments about connections, and ask open-ended questions for students to reflect on. After all the speakers have presented, we have a brief class discussion about the differing ideas, which of them seem most plausible, and why. Students might discuss interactions and relationships in the environment, question connections, and correct each other's misconceptions.

Student interest peaks when we view a five-minute video clip: "Eagles and the Effect of DDT on the Food Chain" (see Internet Resources). Students watch and listen to determine whether their group was able to solve the mystery. The video clip explains how DDT that was sprayed on farmers' fields to control insects was washed by rainfall into streams and rivers, taken up by the plankton, and moved through the food chain to minnows, large fish, and finally the bald eagle. The DDT did not kill adult eagles but affected the hardness of their eggshells. If you do not have access to this video, you can read chapter 4, "The Decline of the Bald Eagle" from *The Bald Eagle Returns* (Patent 2000). Following the video or reading, the groups rearrange their clues to fit the story, and then we watch the video a second time to be sure the clues are in the correct order. Students are asked to draw the chain of clues in their notebooks, showing how DDT moved through the habitat and adding one last fact, the banning of DDT by the U.S. government in 1972.

Last, I have the students write an accompanying paragraph in which they must include a discussion of the clues from their diagram and explain how they are all connected. This writing gives a clear picture of how they grasp habitat relationships. A short list of questions that you might provide for the students to focus their writing follows:

1. What was the original reason DDT was used and by whom?
2. How did the DDT enter the bald eagle's habitat?
3. Explain how DDT moved through the habitat to the bald eagle (use all clues).
4. What was done to stop the DDT from causing more harm to the environment?

Teachers may also want to use a rubric to promote equitable feedback on the notebook portion (adapted from Kopp 2008; see NSTA Connection for rubric). Provide the rubric before the students begin their writing.

Elaboration: Rehabilitators

Discovering that the pesticide DDT was responsible for the alarming decline in the population of bald eagle is only the first step. Once again, I ask the students to put themselves back in the 1960s and ask, "What next"? Once scientists discovered what the problem was, and DDT was banned, would the bald eagle recover on its own? The general consensus of the students is that the eagle would not be able to adapt to thin-shelled eggs and that the DDT would not be removed from the ecosystem fast enough to allow for a natural recovery of the eagle. Next, students wonder how scientists were able to bring eagle populations back. We spend time brainstorming ideas as a class and discussing the possible effects of each. This is a second excellent opportunity to have students consider habitat relationships and human interactions within them. Students' suggestions range from building large cages for eagles to live in to teaching eagles to eat foods other than fish. This is an opportunity to discuss the importance of keeping the eagles wild and free. Many student comments reflect the facts gathered in the activity that create challenges to human help: inaccessible nests that are in the tallest trees often in remote areas; dependency on parents for food for 13 or more weeks; and nesting pairs returning to the same nest year after year to raise their young. How to help a species so well adapted to living in the wild is difficult for many students to fathom.

After accumulating a list of possible solutions, I read from the book by Dorothy Hinshaw Patent, *The Bald Eagle Returns* (2000). Chapter 5 describes why scientists

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Keywords: Mass Extinctions
www.scilinks.org
Enter code: SC011002

had to intervene and how the U.S. government conducted the recovery and restoration of the eagle population. The book does an excellent job of explaining the categories of endangered and threatened species and relating these terms directly to the bald eagle. For many of my students, the book provides an "aha" moment for terms they have heard but never fully understood.

Could this happen again? Could it be happening now? Often questions arise about other species (including humans) that might have been affected by DDT. This is an excellent opportunity for extension activities for individuals or groups to look into other species that were harmed by DDT. Patent (2000) discusses some of the other species that were victims of DDT that the students could investigate. Is there something else that humans do that could be unintentionally affecting another species? Asking these open-ended questions at this point in the lesson leads to important conversations. Students also brought up the puzzling decline in honey bees, another species in our area.

Evaluation: Tying Clues Together

The most concrete assessment of learning are the explanations students write in their notebooks, which I collect and review to assess student understanding of habitat relationships. Writing explanations that tie the clues together is an excellent opportunity for meaningful expository writing and generates a reference that will prove useful in subsequent activities.

I provide feedback on student writing in science notebooks using rubrics or sticky notes. Often students ask whether they can change or fix their writing based on my comments, and I encourage this. This shows that the students value their writing and view the notebook as a meaningful part of the learning process in which they can extend their knowledge (Mintz and Calhoun 2004).

Teacher observation is one of the best assessments of learning. During subsequent classes I often hear "remember the bald eagle" as students compare other environmental problems with the bald eagle activity. This is evidence that students are thinking about interactions and looking for similarities in different situations—they are making meaningful connections. Lessons from the past may be our best way to prevent future harm to species. ■

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Connecting to the Standards

This article relates to the following *National Science Education Standards* (NRC 1996):

Content Standards

Grades K–4

Standard A: Science as Inquiry

- Abilities necessary to do scientific inquiry
- Understanding about scientific inquiry

Standard C: Life Science

- Characteristics of organisms
- Life cycles of organisms
- Organisms and environments

National Research Council (NRC). 1996. *National science education standards.* Washington, DC: National Academies Press.

References

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Mintz, E., and J. Calhoun. 2004. Project notebook. *Science and Children* 42 (3): 30–34.

Patent, D.H. 2000. *The bald eagle returns.* New York: Clarion Books.

Resources

Bailey, J., and D. Burnie. 1992. *Birds: How to watch and understand the fascinating world of birds.* London: Dorling Kindersley.

McConoughey, J. 1983. *The bald eagle.* Mankato, MN: Crestwood House.

Potts, S. 1998. *The bald eagle.* Mankato, MN: Capstone Press.

Wexo, J.B. 1989. *Eagles.* Mankato, MN: Creative Education.

Whitehead, R. 1968. *The first book of eagles.* New York: Franklin Watts.

Internet Resources

Bald Eagle General Facts
www.baldeagleinfo.com/eagle/eagle-facts.html

DDT Label Picture
www.eoearth.org/media/approved/9/9a/Ddt.jpg

Eagles and the Effect of DDT on the Food Chain
http://streaming.discoveryeducation.com

Fast Facts on Bald Eagles
www.wildlifedepartment.com/fastfacts.htm

NSTA Connection

Find a sample rubric and a DDT label at *www.nsta.org/SC1001.*

Title: Integrating Content to Create Problem-Solving Opportunities
Author: Darrin Beigie
Journal: *Mathematics Teaching in the Middle School*
Issue: February 2008, Volume13, Issue 6, pp. 352-360

Article #9

Rationale for Use

Problem Solving is one of the Process Standards in Principles and Standards for School Mathematics, NCTM 2000. The Grades 6-8 Problem Solving Standard (page 256) states that "Problem solving in grades 6-8 should promote mathematical learning. Students can learn about, and deepen their understanding of mathematical concepts by working through carefully selected problems that allow applications of mathematics to other contexts." The author in this article describes how this idea can become a reality in the classroom.

This article describes how traditional exercises can be integrated to form challenging problem–solving situations, to foster higher–order thinking skills and a better understanding and mastery of mathematical content.

Procedure
Session #1

1. Ask participants to read the quote from Principles and Standards for School Mathematics at the beginning of the article: "*Problem Solving is not a distinct topic, but a process that should permeate the study of mathematics and provide a context in which concepts and skills are learned* (NCTM, 2000, p.182)."

- Working in either large or small groups, have participants, based on their experiences, share:
    - What it means to say problem-solving is a process;
    - That problem-solving should provide a context in which concepts and skills are learned; and
    - The challenges presented by the message in the above quote for classroom instruction.

2. Assign Problem 1, Leaning Ladder Problem, Figure 1, page 354 of the article, for participants to work on together (groups of 2 to 4).

- Share with the whole group solutions to the problem along with a discussion of the following:
    - The different strategies they used, even those that did not lead directly to a solution.
    - The mathematical content in this problem.
    - The challenges this problem may present to students.
    - Why did the author considered this a problem-solving task? How does it meet the description of problem-solving stated in the quote discussed in #1?
    - Why did the author say that this problem is the combination of two mathematical procedures, and hence the result is a non-routine problem for most student?

3. Read the section titled: **Integrating Content to Create a Problem Solving Opportunity** on pages 354 – 359, then discuss the following (large or small groups):

- Choose a statement from this Section that you find interesting. Share why you find it so.
- Examine the examples of student's work included in this Section, Figures 2 and 3:
  - How do these solutions compare to what you did?
  - Based on the students' written work, how would you compare these two students' problem-solving abilities? Relate it to the statement in the article from NCTM (2000), " students' problem-solving failures are often due not to a lack of mathematical knowledge but to the ineffective use of what they do know." P.54
  - What questions could you ask the student whose work is included in figure 3, page 355 of the article, to help him or her sort through their ideas to possibly lead to a more successful solution. Is there evidence that this student could integrate content to solve a problem?
- The author says that this article describes, "one seventh-grade teachers' classrooms efforts to integrate traditional exercises from different content areas to form more robust questions that provide genuine problem-solving opportunities for students" P. 352. Share your thoughts on this statement in reference to the problem and the section from the article discussed above.

Session #2

1. Assign Problem 4, Pythagorean Triple Problem, Figure 9, page 358.

   Pythagorean Triplets Problem - Give all examples you can determine of three positive integers satisfying the Pythagorean equation: a^2 + b^2 = c^2

   NOTE: This is an example of an open-ended problem that has infinite solutions. This problem challenges students to find and realize that some problems have more than one solution, and most importantly, asks them to make a generalization to describe these solutions. In this case the generalization can be described by an algebraic equation. It also challenges students to go the next step and justify this generalization.

- Ask participants to solve the problem and as they do consider the following:
  - What is the mathematics children use when doing this problem?
  - Why is this task considered problem-solving? Why is it an open-ended problem?
  - Do you think your students would know that an essential ingredient of problem-solving with open-ended solutions is to make and justify a generalization about the solution. Is it clear in this problem?
  - While students are solving this problem, what clarifying questions might you ask students who are struggling with finding a solution or getting started to find a solution?
- Share different approaches to how participants solved the problem and discuss questions above.

2. Ask participants about the opportunities in their mathematics program for students to respond to open-ended problems.

3. Examine one student's solution to this problem, in Figure 10. Page 358.
- What do you notice about the solution?
- What strategies did the student use?
- Has the student made an appropriate generalization for the solution? Why or why not?
- What question(s) might you ask the student to have him or her consider other Pythagorean triplets, such as, 5, 12, 13.

Next Steps

1. Ask participants to consider assigning at least one of the four problems posed by the author in Figures 11, 12, 13, 14, pages 358 -359 to their students. Then bring to next session student solutions and reflections on the following:
- Why is that problem, chosen from Problem 5 - 8, considered a problem- solving experience rather than just an exercise, as described by the author, page 352? How was it a problem-solving experience for your students?
- What mathematical content is used or explored in more depth in this problem?
- What strategies did the children use to solve the problem?
- What challenges did this problem present to the students?
- Share your role as teacher while students worked on solving this problem. In what way did it present challenges for you?

2. Provide each participant with a copy of the Problem Solving Standard, Grades 6 -8, Principles and Standards for School Mathematics, NCTM, 2000. Ask them to read the Standard, considering the following questions to direct discussion:
- What ideas in this Standard do you find particularly interesting? Why?
- What idea from the Standard has special implications for you when using a problem solving approach to instruction?
- What idea(s) in this Standard would you like to explore further or know more about?
- Share an idea in this Standard, you have not used already, but would like to try in your classroom. What support would you need?
- What are some of the challenges you anticipate in using a problem solving- approach to teaching mathematics, as advocated in the Standard and in the article "Integrating Content to Create Problem-Solving Opportunities", Darrin Beigie, MTMS, Vol 13 No 6 Feb 2008.

3. Read the article Bay-Williams, Jennifer M., and Margaret R. Meyer. "Why Not Just Tell Students How to Solve the Problem?" Mathematics Teaching in the Middle School 10(March 2005):340-41. This article is included in the bibliography of the enhanced article above.
   Suppose at your school you are asked to make a presentation to parents to help them understand why the school is taking a problem solving approach to teaching and learning mathematics. What ideas from this article could you use in your presentation to convince parents or caregivers that their child is more effectively learning and retaining content, along with developing a confidence that they can solve new, unfamiliar problems?

Connections to other NCTM Publications

- Charles, R. I., Lester, F. K., & O'Daffer, P. (1987). *How to Evaluate Progress in Problem Solving.* Reston, VA: National Council of Teachers of Mathematics.
- Clement, L. L., & Bernhard, J. Z. (2005, March). A problem-solving alternative to using key words. *Mathematics Teaching in the Middle School, 10,* 360-365.
- Edwards, B. (2005, August). The thinking of students: Have you lost your marbles? Three creative problem-solving approaches. *Mathematics Teaching in the Middle School, 11,* 18-21.
- Friel et al, S. (2009). *Navigating through Problem Solving and Reasoning in Grades 6-8.* Reston, VA: National Council of Teachers of Mathematics.
- Leitze, A. R., & Mau, S. T. (1999, February). Assessing problem-solving thought. *Mathematics Teaching in the Middle School, 4,* 305-311.
- Lester, F. K., & Charles, R. I. (2003). *Teaching Mathematics through Problem Solving: Prekindergarten- Grade 6.* Reston, VA: National Council of Teachers of Mathematics.
- Malloy, C. E., & Guild, B. (2000, October). Problem solving in the middle grades. *Mathematics Teaching in the Middle School, 6,* 105-108.
- Pugalee, D. K., & Malloy, C. E. (1999, February). Teachers' action in community problem solving. *Mathematics Teaching in the Middle School, 4,* 296-300.
- Schoen, H. L., & Charles, R. I. (2003). *Teaching Mathematics through Problem Solving: Grades 6-12.* Reston, VA: National Council of Teachers of Mathematics.
- Wallace, A. H. (2007, May). Anticipating student responses to improve problem solving. *Mathematics Teaching in the Middle School, 12,* 504-511.

Posted on April 8, 2010

Article #10

Why Is Teaching With Problem Solving Important to Student Learning?

PROBLEM solving plays an important role in mathematics and should have a prominent role in the mathematics education of K–12 students. However, knowing how to incorporate problem solving meaningfully into the mathematics curriculum is not necessarily obvious to mathematics teachers. (The term "problem solving" refers to mathematical tasks that have the potential to provide intellectual challenges for enhancing students' mathematical understanding and development.) Fortunately, a considerable amount of research on teaching and learning mathematical problem solving has been conducted during the past 40 years or so and, taken collectively; this body of work provides useful suggestions for both teachers and curriculum writers. The following brief provides some directions on teaching with problem solving based on research.

What kinds of problem-solving activities should students be given?

Story or word problems often come to mind in a discussion about problem solving. However, this conception of problem solving is limited. Some "story problems" are not problematic enough for students and hence should only be considered as exercises for students to perform. For example, students may be asked to find the perimeter of a polygon, given the length of each side. They can mindlessly add these numbers and get the answer without understanding the concept of perimeter and the problem situation. However, some nonstory problems can be true problems, such as those found, for example, while playing mathematical games.

In general, when researchers use the term *problem solving* they are referring to mathematical tasks that have the potential to provide intellectual challenges that can enhance students' mathematical development. Such tasks—that is, problems—can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity (Hiebert & Wearne, 1993; Marcus & Fey, 2003; NCTM, 1991; van de Walle, 2003). Research recommends that students should be exposed to truly problematic tasks so that mathematical sense making is practiced (Marcus & Fey, 2003; NCTM, 1991; van de Walle, 2003). Mathematical problems that are truly problematic and involve significant mathematics have the potential to provide the intellectual contexts for students' mathematical development. However, only "worthwhile problems" give students the chance to solidify and extend what they know and stimulate mathematics learning. That said, what is a worthwhile problem? Regardless of the context, worthwhile tasks should be intriguing and contain a level of challenge that invites speculation and hard work. Most important, worthwhile mathematical tasks should direct students to investigate important mathematical ideas and ways of thinking toward the learning goals (NCTM, 1991). Lappan and Phillips (1998) developed a set of criteria for a good problem that they used to develop their middle school mathematics curriculum (Connected Mathematics), and there has been some research supporting the effectiveness of this curriculum for fostering students' conceptual understanding and problem solving (Cai, Moyer, Wang, & Nie, in press). Although there has been no research focusing specifically on the effectiveness of this set of criteria, the fact that the curriculum as a whole has been shown to be effective suggests that teachers might want to attend to this set in choosing, revising, and designing problems. See the following worthwhile-problem criteria:

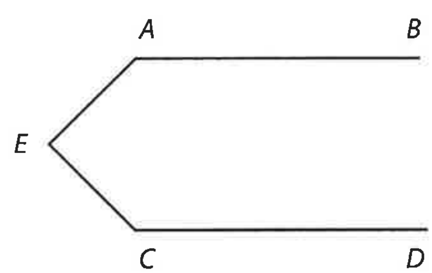
1. The problem has important, useful mathematics embedded in it.
2. The problem requires higher-level thinking and problem solving.
3. The problem contributes to the conceptual development of students.
4. The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.
5. The problem can be approached by students in multiple ways using different solution strategies.
6. The problem has various solutions or allows different decisions or positions to be taken and defended.
7. The problem encourages student engagement and discourse.
8. The problem connects to other important mathematical ideas.
9. The problem promotes the skillful use of mathematics.

10. The problem provides an opportunity to practice important skills.

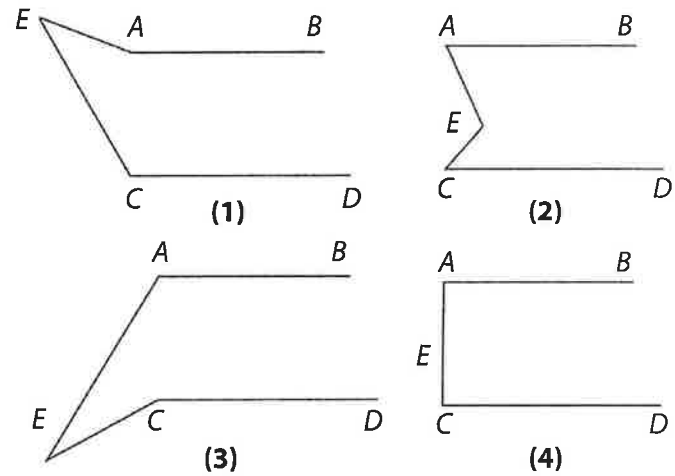
Of course, it is not reasonable to expect that every problem that a teacher chooses should satisfy all 10 criteria; which criteria to consider should depend on a teacher's instructional goals. For example, some problems are used primarily because they provide students with an opportunity to practice a certain skill (criterion 10), say, solving a proportion, whereas others are used primarily to encourage students to collaborate with one another and justify their thinking (criteria 6 and 7). But researchers and curriculum developers alike tend to agree that the first four criteria (important mathematics, higher-level thinking, conceptual development, and opportunity to assess learning) should be considered essential in the selection of all problems. Indeed, these four can be regarded as the sine qua non of the criteria. The real value of these criteria is that they provide teachers with guidelines for making decisions about how to make problem solving a central aspect of their instruction.

The role of teachers is to revise, select, and develop tasks that are likely to foster the development of understandings and mastery of procedures in a way that also promotes the development of abilities to solve problems and reason and communicate mathematically (NCTM, 1991). The following example illustrates how a teacher can modify a standard textbook problem in a way that both engages students in learning important mathematics (criterion 1) and also enhances the development of their problem-solving abilities (criteria 2, 3, 4, and 5).

EXAMPLE. Original problem (Cai & Nie, 2007) (Grades 9–11): In the figure below, segment AB is parallel to segment CD. Show that the sum of the measures of $\angle A$, $\angle E$, and $\angle C$ is 360°.



This problem might be found in any standard textbook. It clearly involves important mathematics, but in its present form, criteria 2, 3, 4, and 5 are not as clearly included. By making a quite modest revision, we can open up the problem and by doing so raise the cognitive demand (criterion 2) and also satisfy criteria 3 and 4: *Revised problem:* What is the sum of the measures of $\angle A$, $\angle E$, and $\angle C$? In addition, we might ask students to find the sum of the three angles in different ways and make generalization of the problem by asking: What is the sum of the three angle measures if point E is at different locations (as shown in the figures below)?



This example illustrates that modifying problems that already exist in textbooks is often a relatively easy thing to do but increases the learning opportunity for students. Indeed, the revised problems need not be complicated or have a fancy format. Readers may also see (Butts, 1980) how to revise a problem to be more problematic so that the learning opportunity for students is increased.

Should problem solving be taught as a separate topic in the mathematics curriculum or should it be integrated throughout the curriculum?

There is little or no evidence that students' problem-solving abilities are improved by isolating problem solving from learning mathematics concepts and procedures. That is, the common approach of first teaching the concepts and procedures, then assigning one-step "story" problems that are designed to provide practice on the content learned, then teaching problem solving as a collection of strategies such as "draw a picture" or "guess and check," and finally, if time, providing students with applied problems that will require the mathematics learned in the first step (Lesh & Zawojewski, 2007, p. 765), is not supported by research. In fact, the evidence has mounted over the past 30 years that such an approach does not improve students' problem solving to the point that today no research is being conducted with this approach as an instructional intervention (e.g., Begle, 1973; Charles & Silver, 1988; Lester, 1980; Schoenfeld, 1979). The implication of this change in perspective is that if we are to help students become successful problem solvers, we first need to change our views of problem solving as a topic that is added onto instruction after concepts and skills have been taught. One alternative is to make problem solving an integral part of mathematics learning. This alternative, often called teaching through problem solving, adopts the view that the connection between problem solving and concept learning is symbiotic (Lambdin, 2003): Students learn and understand mathematics through solving mathematically rich problems and problem-solving skills are developed through learning and understanding mathematics concepts and procedures (Schroeder & Lester, 1989).

In teaching through problem solving, learning takes place during the process of attempting to solve problems in which relevant mathematics concepts and skills are embedded (Lester & Charles, 2003; Schoen & Charles, 2003). As students solve problems, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify their ideas in ways that they feel are convincing. The learning environment of teaching through problem solving provides a natural setting for students to present various solutions to their group or class and learn mathematics through social interactions, meaning negotiation, and reaching shared understanding. Such activities help students clarify their ideas and acquire different perspectives of the concept or idea they are learning. Empirically, teaching mathematics through problem solving helps students go beyond acquiring isolated ideas toward developing increasingly connected and complex system of knowledge (e.g., Cai, 2003; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cobb et al. 1991; Hiebert & Wearne, 1993; Lambdin, 2003). The power of problem solving is that obtaining a successful solution requires students to refine, combine, and modify knowledge they have already learned.

It is important to point out that we are not saying that every task that students encounter must be problematic. If the goal of a lesson is to develop and master certain skills, some exercises are necessary. In addition, as we indicated before, teachers may modify existing less problematic problems to be "true" problems.

How can teachers orchestrate pedagogically sound, active problem solving in the classroom?

Picking the problem or task is only one part of teaching with problem solving. There is considerable evidence that even when teachers have good problems they may not be implemented as intended. Students' actual opportunities to learn depend not only on the type of mathematical tasks that teachers pose but also on the kinds of classroom discourse that takes place during problem solving, both between the teacher and students and among students. Discourse refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in instructional tasks. Considerable theoretical and empirical evidence exists supporting the connection between classroom discourse and student learning. The theoretical support comes from both constructivist and sociocultural perspectives of learning (e.g., Cobb, 1994; Hatano, 1988; Hiebert et al., 1997). As students explain and justify their thinking and challenge the explanations of their peers and teachers, they are also engaging in clarification of their own thinking and becoming owners of "knowing" (Lampert, 1990). The empirical evidence supporting the positive relationships between teachers' asking high-order questions and students' learning can be found in the work of Hiebert and Wearne (1993) and of Redfield and Rousseau 1981).

Then what is considered to be desirable discourse in mathematics teaching? To explore this question, let us compare the two teaching episodes shown below involving seventh-grade teachers and their students (Thompson, Philipp, Thompson, & Boyd, 1994). The teachers presented the following problem to their classes:

> At some time in the future John will be 38 years old. At that time he will be 3 times as old as Sally. Sally is now 7 years old. How old is John now?

Teaching Episode 1

T: Let's talk about this problem a bit. How is it that you thought about it?

S1: I divided 38 by 3 and I got 12 2/3. Then I subtracted 7 from 12 2/3 and got 5 2/3. [Pause] Then I subtracted that from 38 and got 32 1/3. [Pause] John is 32 1/3.

T: That's good! [Pause] Can you explain what you did in more detail? Why did you divide 38 by 3?

S1: [Appearing puzzled by the question, S1 looks back at her work. She looks again at the original problem.] Because I knew that John is older—3 times older.

T: Okay, and then what did you do?

S1: Then I subtracted 7 and got 5 2/3. [Pause] I took that away from 38, and that gave me 32 1/3.

T: Why did you take 5 2/3 away from 38?

S1: [Pause] To find out how old John is.

T: Okay, and you got 32 1/3 for John's age. That's good! [Pause]

Teaching Episode 2

T: Let's talk about this problem a bit. How is it that you thought about the information in it?

S1: Well, you gotta start by dividing 38 by 3. Then you take away . . .

T: [Interrupting] Wait! Before going on, tell us about the calculations you did, explain to us why you did what you did. (Pause) What were you trying to find?

S1: Well, you know that John is 3 times as old as Sally, so you divide 38 by 3 to find out how old Sally is.

T: Do you all agree with S1's thinking?

[Several students say "Yes"; others nod their heads.]

S2: That's not gonna tell you how old Sally is now. It'll tell you how old Sally is when John is 38.

T: Is that what you had in mind, S1?

S1: Yes.

T: [To the rest of the class] What does the 38 stand for?

S2: John's age in the future.

T: So 38 is not how old John is now. It's how old John will be in the future. [Pause] The problem says that when John gets to be 38 he will be 3 times as old as Sally. Does that mean "3 times as old as Sally is now" or "3 times as old as Sally will be when John is 38"?

[Several students respond in unison, "When John is 38."]

T: Are we all clear on S2's reasoning? [Pause]

There are a number of similarities between the two teaching episodes that Thompson and colleagues analyzed. For example, both teachers opened their lessons with the same problem and with similar instructions. Both teachers pressed their students to give rationales for their calculation procedures. However, the two teaching episodes differed significantly in terms of how the teachers led the classroom discussion. For example, students in Teaching Episode 2 began to give explanations that were grounded in conceptions of the situation (i.e., in making sense of the situation presented in the problem). By contrast, the explanations given by students in Teaching Episode 1 remained strictly procedural. In addition, Teacher 1 was less persistent than Teacher 2 in probing the students' thinking. He accepted solutions consisting of calculation sequences. However, Teacher 2 persistently probed students' thinking whenever their responses were cast in terms of numbers and operations. The analysis clearly shows that mathematical tasks can be implemented differently, depending on the nature of classroom discourse (Knuth & Peressini, 2001; Sherin, 2000; Silver & Smith, 1996; Thompson et al., 1994).

There are a number of factors that can influence the implementation of worthwhile problems in classrooms (e.g., Henningsen & Stein, 1997). One of the predominant factors is the amount of time allocated to solving and discussing the problem. For example, Rowe (1974) found that the mean time that teachers waited between asking a question and, if no answer was forthcoming, intervening again was only 0.9 seconds. A wait time of less than one second prevented most students from taking part in the classroom discussion. Consequently, it is no wonder that many students believe that every problem should be solvable with little or no thinking (Lesh & Zawojewski, 2007). Another important barrier to meaningful problem solving experiences is that teachers often remove the challenges of a mathematical task by taking over the thinking and reasoning and telling students how to solve the problem. There is considerable evidence that many U.S. mathematics teachers think that they have the responsibility to remove the challenge (and the struggle) for their students when they are engaged in problem solving. In her study of eighth-grade students who were part of the Third International Mathematics and Science Study (TIMSS), Smith (2000) found that U.S. teachers almost always intervened to show students how to solve the problems they had been asked to solve, leaving the mathematics they were left to do rather straightforward. This stands in direct contrast to teachers in Germany and Japan, who allowed students much greater opportunities to struggle with the more challenging parts of the problems. Productive struggle with complex mathematical ideas is crucial to learning during problem solving. Finally, teachers are also responsible for listening carefully to students' ideas and asking them to clarify and justify their ideas orally and in writing, as well as monitoring their participation in discussions and deciding when and how to encourage each student to participate. The questions that teachers ask are also critical for orchestrating sound classroom discourse (Rasmussen, Yackel, & King, 2003; Stephan & Whitenack, 2003).

Conclusion

To help students become successful problem solvers, teachers must accept that students' problem-solving abilities often develop slowly, thereby requiring long-term, sustained attention to making problem solving an integral part of the mathematics program. Moreover, teachers must develop a problem-solving culture in classroom to make problem solving a regular and consistent part of one's classroom practice. Students must also buy into the importance of regularly engaging in challenging activities (Lester, 1994; Willoughby, 1990).

Developing students' abilities to solve problems is not only a fundamental part of mathematics learning across content areas but also an integral part of mathematics learning across grade levels. Beginning in preschool or kindergarten, students should be taught mathematics in a way that fosters understanding of mathematics concepts and procedures and solving problems. In fact, there is strong evidence that even very young students are quite capable of exploring problem situations and inventing strategies to solve the problems (e.g., Ben-Chaim et al., 1998; Cai, 2000; Carpenter et al., 1998; Kamii & Housman, 1989; Maher & Martino, 1996; Resnick, 1989). However, students cannot become successful problem solvers overnight. Helping students become successful problem solvers should be a long-term instructional goal, so effort should be made to reach this goal at every grade level, in every mathematical topic, and in every lesson.

Research clearly suggests that problem solving should not be taught as a separate topic in the mathematics curriculum. In fact, research tells us that teaching students to use general problem-solving strategies has little effect on their success as problem solvers. Thus, problem solving must be taught as an integral part of mathematics learning, and it requires a significant commitment in the curriculum at every grade level and in every mathematical topic. In addition to making a commitment to problem solving in the mathematics curriculum, teachers need to be strategic in selecting appropriate tasks and orchestrating classroom discourse to maximize learning opportunities. In particular, teachers should engage students in a variety of problem-solving activities: (a) finding multiple solution strategies for a given problem, (b) engaging in mathematical exploration, (c) giving reasons for their solutions, and (d) making generalizations. Focusing on problem solving in the classroom not only impacts the development of students' higher-order thinking skills but also reinforces positive attitudes. Finally, there is no evidence that we should worry that students sacrifice their basic skills if teachers focus on developing their mathematical problem-solving skills.

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Judith Reed Quander, Series Editor

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