Being mathematical versus mathematical be-ing: situated activity, interactive emergence and mathematics education[1]

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Abstract

Although most work in artificial intelligence has been focused on reproducing the results of human thought, without paying any attention to whether the processes used by computers bear any relation to human cognition, over recent years there has been increasing interest in computer models of thought that, as well as producing expert performance, exhibit learning that is analogous to human learning. This new literature challenges the Cartesian dualism of mind and body, and stresses the embodied nature of human cognition.

Drawing on the work of Clark (1997) and Hendriks-Jansen (1997) and Karmiloff-Smith (1992) this paper explores the relevance of computer models of human thought for education (and specifically for mathematics education) and will deal in particular with the issue of whether internal representations are necessary for expert performance.

The paper concludes that conventional ways of looking at students' work in mathematics classrooms in terms of heuristics are *post hoc* impositions onto the phenomena rather than being intrinsic features of the phenomena themselves.

Introduction

It is almost a truism that cultures have built models of the mind that reflect prevailing explanatory mechanisms for other phenomena. For Descartes, mind was to be thought of as a spiritual entity, created by a supernatural deity. In Victorian times, explanations were often cast in terms of machinery and mechanical systems. More recently, since the advent of high-speed electronic information processors, cognitive scientists have pursued models of human thought based on metaphors such as 'information processing', 'storage', and 'short-term working memory'.

Such metaphors can be useful, but they can also impede progress. Early attempts at mechanical flight by humans were unsuccessful because they tried to copy avian flight, typically with flapping wings. It was only when people stopped trying to fly like birds, and started to try to achieve the same ends by different means, that mechanical flight became possible (Papert, 1980). In a similar vein, most of the progress that has been made in artificial intelligence (AI) over the last thirty years has been due to an abandonment of the pretence that the processes used in AI are in any way analogous to processes used by humans to produce the same ends.

One result of this, and one that, because of these differences in approach, we should not, perhaps, find surprising, is that substantial progress has been made in getting computers to display 'high-level' skills such as playing chess, while relatively 'low-level' skills, such as face-recognition have proved surprisingly intractable.

One response to this has been an increased interest in artificial models of human thought that not only reproduce the results of human thought, but do so by means that are analogous to the ways in which humans produce the same results.

It is the contention of this paper that most of current thinking in mathematics education is based too heavily on a view of the human mind as some kind of central processor, with a central controller to activate various plans or 'scripts'. If, as I shall argue, this model does not describe well the processes of human thought, then there is a very real danger that the prescriptions for mathematics teaching that arise from such views are misleading, or just plain wrong.

In this paper I have not tried to spell out in detail a carefully reasoned argument that leads inevitably to my conclusion—indeed, it is part of my argument that, within the limits of

current neuroscience, no such demonstration is possible, and in any case, the views that I propose in this paper are as yet still rather incoherent. However, I am convinced that there is 'a case to answer'—that there is something fundamentally wrong with the ways of thinking about the mathematical activity of learners that currently predominate, and the purpose of this paper is to begin a discussion that may lead at least to clarification of the issues involved.

The embodied mind

Crucial to recent work in psychology has been the realisation that human minds were not designed, but rather evolved (Barkow, Cosmides, & Tooby, 1992; Crawford & Krebs, 1998; Stevens & Price, 1996), and in particular, for most of the period of its evolution, evolved in order to regulate the activity of a body. This may seem rather obvious, which makes all the more inexplicable the emphasis in much psychology on the mind as a disembodied processor.

One of the most important realisations arising from this work is that the requirement that each evolutionary adaptation must confer an immediate advantage for it to be selected (Dawkins, 1986) means that evolutionary solutions are rarely optimally designed (Gould, 1980). A good example of this is the rather perverse arrangement in mammals in which the left side of the brain controls the movement of the right side of the body, which results in all kinds of complexities in the organisation of minds and bodies. The reason for this is that the earliest amphibian eyes had lenses. In the absence of a lens, the image on a retina (or its equivalent) of an object moving from left to right would also move from left to right. However, with a lens, the retinal image moves from right to left. In order to track the object, the eye (or the head) needs to swivel to the right, requiring a contraction of muscles on the right side of the organism. But the stimulus for this action (the retinal image of the object being tracked) is on the *left* side of the retina, so the neural pathway needs to go from the left side of the eye to the right side of the body (Klawans, 1996).

Adopting an evolutionary perspective on psychology therefore requires us to look at what minds have actually evolved to do, not in the modern world, but in the 'environment of evolutionary adaptedness' (Bowlby, 1969).

One of the most fruitful areas of work in this regard has been the psychology of vision and perception. The earliest models of vision were based on 'photographic' metaphors, with the eyes as receptors and the brain as an essentially passive processor of visual information. However, there is increasing evidence that vision involves a high degree of interaction between the eye and the brain. For example, even in such primitive creatures as horseshoe crabs, it appears that some visual processing actually takes place in the retina (Passaglia, Dodge, Herzog, Jackson, & Barlow, 1997).

What is perhaps even more surprising is the extent to which what we actually see is a result of our previous experiences. In the well-known optical illusion shown in figure 1, generally known as the Müller-Lyer illusion after its discoverer, the German psychologist Franz Carl Müller-Lyer, the two vertical line segments are exactly the same length. However, the left-hand vertical appears to be much shorter than the one on the right. The most convincing explanation of this has been provided by Richard Gregory who suggests that the illusion is caused by the visual processing system's interpretation of the left hand figure as an external corner of an object, while that on the right is an internal corner (Gregory, 1966). Since the visual perception system 'knows' that objects that are further away are smaller, it compensates for this by creating a *perception* of the internal corner as larger, because it must be further away. As Hundert (1995 p211) observes:

> It is tempting to call this phenomenon of the accommodation of our plastic visual input analyzers to the realities of depth perspective and size constancy in our world a "natural phenomenon"—except for the simple fact that such rectilinear lines do not exist *in nature!* (emphasis in original)

And indeed, it has been found that people raised in cultures who do not build rectilinear structures do not experience the Müller-Lyer illusion (Deregowsky, 1974). Even something as basic as what we see, therefore, develops only by an interaction between the embodied mind and its environment.

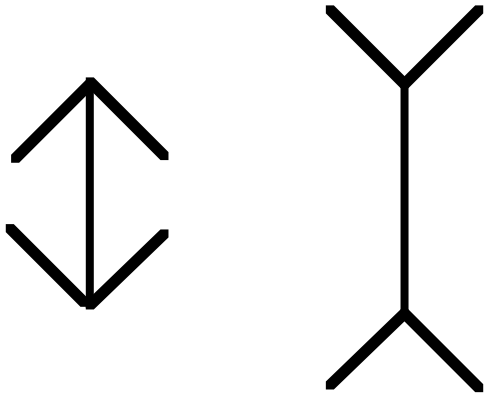

Figure 1: the Müller-Lyer illusion

A similar effect is observed if one holds both one's hands in front of one's face, one hand being held twice as far away from the eyes as the other. The laws of physics tell us that the nearer hand should be seen as twice as large (and as having four times the area) of the further hand, but for most people, this is not what is observed

A further example is provide by the effect of 'colour constancy' whereby objects do not appear to change colour as the lighting conditions vary, or as the object moves to a region whereby the ambient lighting conditions are different. Again, the laws of physics tell us that the actual wavelength of the light impacting the retina changes, but this is not perceived as a change in colour. The mind's *theoretical* model—that the object has not changed colour—overrides the sense data.

What these three examples illustrate very clearly is that input-process-output models of the human mind do not account for much of the evidence that we now have about the extent to which human cognition is a result of the interaction between the individual and the environment, and although this has been recognised in the work arising from Soviet activity theory (Christiansen & Walther, 1986; Leont'ev, 1972) and the 'rediscovery' of the work of Dewey (Garrison, 1995), there is still, as Jean Lave has often argued (see for example, Lave, 1996), a tendency to regard cognitive activity as taking place entirely 'in the mind'.

This is not to say that there is no such thing as individual minds. The absurdity of such a position is nicely exemplified by the following (apparently true) anecdote. A driver had been involved in an accident after driving the wrong way round a roundabout. When asked, "Haven't you ever driven round a roundabout before?", the driver replied, "Yes, but not *this* roundabout". We find the driver's reply amusing or absurd precisely because we find it incomprehensible that an individual would not be able to apply their experience of previous roundabouts to an unfamiliar roundabout. However, while not wishing to deny the existence of individual minds, I do want to argue that it makes no sense to attempt to look for explanations of human cognition either 'inside' or 'outside' the body. It seems to me that useful explanations of human cognition will be developed only if we look on cognition as an *emergent* phenomenon.

The idea of an emergent activity is very clearly illustrated by the work of Maja Mataric—a researcher in the field of artificial intelligence who has been developing robots that can be said to act in a way that might be regarded as intelligent. The following account is based on the description given by Hendriks-Jansen (1997).

One of Mataric's robots has 12 (not very accurate) sensors distributed around its perimeter, and has four different behaviours: 'stroll', 'avoid', 'align', and 'correct'. 'Stroll' and 'avoid' use the two 'forward' pointing sensors, and partition the space in front of the robot into two zones—a 'danger' zone extending 0.3 metres in front of the robot, and a 'safe' zone extending a further 0.3 metres beyond that. Detecting an object in the danger zone triggers the 'stroll' behaviour, which involves backing up, and the detection of an object in the 'safe' zone activates the 'avoid' behaviour which involves turning 30° away from the object. If the left-forward sensor detects an object, the robot turns to the right, and if the right-forward sensor detects an object, the robot turns to the left. In order to avoid the problem of Buridan's ass

(Fernández-Armesto, 1997 p83), if an object is detected by both sensors simultaneously, the robot turns to the left. In the absence of any objects ahead, the robot travels directly forward for a set distance.

The 'align' behaviour uses the two pairs of lateral sensors, which operate with an imaginary "edging distance" of 0.8 metres, and the two rear-lateral sensors. If an object is detected by the rear-lateral sensor on (say) the right side, but not by the two right lateral sensors, then the robot turns 30° to the right. In other words, if the robot is heading away from a surface, it will turn towards that surface. The 'correct' behaviour involves just the two pairs of lateral sensors—if the rearmost of (say) the two right-lateral sensors senses an object while the foremost does not, then the robot turns 30° to the right. When placed near a wall, the robot then follows a path that first brings the robot close to the wall, and then continues on a track roughly parallel to the wall. At different times, with only very slightly differing starting points, the robot will follow the wall in different ways. Indeed, due to the poor resolution of the sensors, the robot may follow a different path even when started exactly at the same point.

The 'wall-following' behaviour that we observe is in no sense desired or purposive. The robot is just doing what it does. It has no 'representation' of the wall it is following, nor of the path it should follow. Indeed, as Hendriks-Jansen observes, "if there is any 'representation' of walls, it is only that produced by the robot's actual behaviour" (Hendriks-Jansen, 1997 p143). The 'wall-following behaviour' is an imposition by us onto what the robot is doing. What we call 'wall-following' is an emergent activity, created by the interaction of the robot with its environment.

The importance of the interaction between the individual and the environment is clearly illustrated by the problem of finding our way through a reasonably, but not too, familiar route. We have all had the experience of knowing the way to a particular destination, in the sense of being able to find one's way there, while at the same time, not knowing the way there well enough in order to be able to describe it to someone else. This is because our ability to act in a particular setting is not (at this stage) based on an internal representation of the route we wish to follow, but is instead the result of an interaction between the mind and the environment. Because these 'visually mediated sequences' (Davis, 1984) are the result of a series of attunements to the constraints and affordances (Gibson, 1979) provided by the environment, without them, we simply do not know what we will do when we get there. When we do get there, however, the right sequences of actions is cued—again a matter of 'being there' (Clark, 1997). Similar arguments account well for the inability of some people to think through a mathematical problem without a pencil in their hand, even though the pencil is not used to write anything.

Now at some point, with our familiar journey, we do begin to build explicit representations of the journey, so that we are, after a while, able to describe the route to others—a process that Karmiloff-Smith describes as representational redescription (Karmiloff-Smith, 1992). She argues that this representational redescription takes place spontaneously, and at increasingly explicit levels, so that the initial redescriptions, which are not accessible to consciousness, are subsequently augmented by redescriptions that are accessible to consciousness but not to verbal report and then later by redescriptions that are accessible to verbal report.

Now while there is no doubt that such a progression exists in some aspects of human development, it seems to me that this process is not inevitable. For example, most speakers of English speak perfectly grammatical English without any explicit knowledge of grammar. There is no doubt that explicit representations are useful for teachers, but there is very little evidence that teaching *about* grammar improves people's ability to write grammatically any more effectively than becoming immersed in the grammar of the language through writing and reading. Just like Mataric's robot, the learner becomes competent not by building internal images or representations of the object, but by learning how to interact with the object (in this case the language).

Karmiloff-Smith also argues that the higher-level representational redescriptions function as higher-level control strategies that then are used to organise lower-level strategies, and in

some aspects of learning this may well be the case. However, in other aspects, it seems to me at least as likely that the process of representational redescription that Karmiloff-Smith describes is *independent* of performance. The fact that these redescriptions are *available* as high-level control strategies does not mean that they are *used* as such. Indeed, it seems to me that even when high-level redescriptions are available (and are essential if we are to describe the route to others), they need not be—and I would argue generally are not—used when the person is actually making the familiar journey. The existence of a higher-level redescription is necessary if the cues for action are not present, but when they are, then it seems to me much more likely that the activity is carried out at lower level, and not only without conscious control, but without any *awareness* of conscious control.

What has all this to do with mathematics education? When Georg Polya published 'How to solve it' over forty years ago (Polya, 1945), it was intended as an approach to teaching problem-solving—in other words a *pre*scription for teaching problem-solving. However, the heuristics that Polya mentioned—understand the problem, devise a plan, carry it out, and look back—were devised primarily in order simply to organise problem-solving, and to bring out particular points. As it clear from Polya's subsequent work (Polya, 1962; Polya, 1965), competence at solving problems can be achieved only by solving lots of problems.

Subsequently, as more and more work has focused on general strategies, it seems that what were originally called heuristics (ie devices intended for the purposes of *teaching*) came to be regarded as general strategies. In effect, what had been intended as a means of accelerating students towards mathematical competence came to be regarded as constitutive of that competence.

In some ways, perhaps this was inevitable. Mathematics is a demanding subject; not just hard, but requiring a kind of way of thinking that does not appear to be natural for human minds, and anything that could make mathematics easier was bound to be very attractive. There is evidence that teaching generalised strategies can improve performance (Schoenfeld, 1992), but this appears to be at the expense of the validity of that activity. Teaching school students to solve problems by running through a list of general strategies may improve their problem-solving performance in the short term, but it is my contention that this has nothing to do with expert performance. A 'higher-order thinking' approach to teaching would concentrate on a small number of powerful, general purpose strategies that could be used in a variety of situations, but the evidence is that this is not how experts work.

To illustrate this, consider the following mathematical activity which is used in the television programme 'Countdown'. The contestants are given a set of numbers, and, by using the four arithmetic operations of addition, subtraction, multiplication and division, are required to generate an arithmetical expression to the value of a given target number. For example, the given numbers might be 20, 11, 2, 5 and 3, with a target number of 105. There are many relatively general strategies that can be developed for solving problems such as these, but the most interesting feature of such problems for the purposes of this paper is that expert performance does not appear to use such general strategies. In fact most people use surprisingly specialised strategies that only work with the specific constraints and affordances of this particular problem. A typical approach relies on working forwards from the givens and backwards from the goal *at the same time.* It is important to note that this does not appear to be an iterative process, but rather, for most people with a reasonable repertoire of number facts the fact that 5x20 is 100, which is close to the target number, seems to leap unbidden into consciousness.

The concern to describe what students do when they solve mathematical problems has led us to label a whole range of things that when done, lead us to conclude that students are 'being mathematical'. However, as I have argued in this paper, it is a mistake to regard the general strategies that we, as relatively expert mathematical practitioners, project onto the mathematical activities of others as constitutive of those activities. If we are to develop in young people the ability to move towards capability as mathematicians, then we should spend less time on projecting our ideas about what it means to 'be mathematical', and more time 'being there' in the mathematical situation—mathematical be-ing.

Note

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